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A Note on Chambers' Method

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Abstract. A correction is given for one of Chambers' second-order iteration formulae. It is shown that composition of the secant method with itself exhibits a convergence exponent of 2.414, whereas composition of the iteration function with itself yields an exponent of 2.831.

The improved second-order iteration schemes presented recently by Ll. G. Chambers [1] yield estimates of isolated zeros of $f(x)$ which exhibit superquadratic convergence exponents. We wish to point out an error in that paper: namely that Eq. (2.10) which was given as

$$X_{n+1} = \frac{X_{n-1} Y_n^2}{(Y_{n-1} - Y_n)^2} + \left(\frac{Y_n}{Y_n - Y_{n-1}}\right) \cdot \left(\frac{Y_n - X_n Y_n'}{Y_n'}\right)$$

should have read

$$X_{n+1} = \frac{Y_n(X_{n-1} Y_n - X_n Y_{n-1})}{(Y_n - Y_{n-1})^2} + \left(\frac{Y_{n-1}}{Y_n - Y_{n-1}}\right) \cdot \left(\frac{Y_n - X_n Y_n'}{Y_n'}\right). \tag{1}$$

The convergence exponent of 2.414 remains valid however.

In a follow-up note on Chambers' method, M. G. Cox [2] pointed out that "an iteration function constructed by composition of the secant method with itself also requires just two function evaluations per iteration and has a convergence exponent of $(1.618)^2 = 2.618$". We believe this remark to be in error for the reasons given below.

Composition of the secant method with itself would generate the two-stage iteration: $\phi(X_{n-1}, X_n) = X_n^*$; $\phi(X_n, X_n^*) = X_{n+1}$. The first stage has an associated error [3] of $\epsilon_n^* = K_0 \epsilon_{n-1} \epsilon_n$ while the second stage is governed by $\epsilon_{n+1} = K_1 \epsilon_n \epsilon_n^*$. Therefore

$$\epsilon_{n+1} = K \epsilon_{n-1} \epsilon_n^2. \tag{2}$$

Note that this recurrence relation is of the very same form as that given by Chambers in connection with iteration formulae of the Wegstein type when $g = 1$; the convergence exponent in such a case was 2.414. It is incorrect to merely square the convergence exponent of a single secant method step since this would ignore the role of X_n^*. Cox's figure of 2.618 is thus overly optimistic although we concede that the practical significance of such a small difference would probably be nil.

Cox also states that composition of the iteration function with itself yields "a convergence exponent of $(1.839)^2 = 3.382$, which is superior to all those derived

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by Chambers". This method would involve the two-stage procedure: $\phi(X_{n-2}, X_{n-1}, X_n) = X_n^*$; $\phi(X_{n-1}, X_n, X_n^*) = X_{n+1}$. Two new function evaluations per iteration are required—that is $f(X_n)$ and $f(X_n^*)$. The first and second stage errors are then $\epsilon_n^* = K_0\epsilon_{n-2}\epsilon_{n-1}\epsilon_n$ and $\epsilon_{n+1} = K_1\epsilon_{n-1}\epsilon_n\epsilon_n^*$. Therefore,

$$\epsilon_{n+1} = K\epsilon_{n-2}\epsilon_{n-1}^2\epsilon_n^2. \tag{3}$$

Following Chambers, we assume $\epsilon_{n+1} = M\epsilon_n^{\mu}$ and so $\mu = 1/\mu^2 + 2/\mu + 2$, the appropriate root of which is 2.831. We conclude therefore that the convergence exponent arising, when the iteration function is cycled on itself, is 2.831 and not 3.382 as given by Cox. Hence, supercubic performance does not occur.

The relative merits of the various methods are clearly revealed by the convergence rates *per function evaluation* [4]: secant [1.618]; selfcomposed secant [$(2.414)^{1/2} = 1.554$]; quadratic inverse interpolation [1.839]; selfcomposed quadratic inverse interpolation [$(2.831)^{1/2} = 1.683$]; Chambers' first method [$2^{1/2} = 1.414$]; Chambers' second method [$(2.414)^{1/2} = 1.554$]; Chambers' third method [$(2.732)^{1/2} = 1.653$].

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1. LL. G. CHAMBERS, "A quadratic formula for finding the root of an equation," *Math. Comp.*, v. 25, 1971, pp. 305–307. MR **45** #4625.
2. M. G. COX, "A note on Chambers' method for finding a zero of a function," *Math. Comp.*, v. 26, 1972, p. 749.
3. D. M. YOUNG & R. T. GREGORY, *A Survey of Numerical Mathematics*, Vol. I, Addison-Wesley, Reading, Mass., 1972, pp. 150–153.
4. We are indebted to the referee for suggesting this final comparison.